Technical Notes

Characteristic Roots for Donnell's Equations with Torsion

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Nomenclature

a = radius of cylindrical shell
C = parameter = $(1 + \Lambda^2)^{1/2}$
E = elasticity modulus
h = thickness
i = $(-1)^{1/2}$
j = index
n = circumferential harmonic, a positive integer
p = complex-valued root
p_0 = complex-valued root for zero prestress
p^* = $(p/a)(\epsilon^*)^{1/2}$
W = const
w = lateral displacement ratio $\equiv w^*/a$
w^* = lateral displacement
x^* = axial coordinate = $ax(\epsilon^*)^{1/2}$
y^* = circumferential coordinate = $a\phi^*$
Z = complex variable
α = complex number
β = complex number = $\Lambda/4\bar{p}_0{}^2$
β_{0j} = see Eq. (21)
ϵ = small number = $\rho_1/\bar{p}_0$
ϵ^* = $h/2a[3(1 - \nu^2)]^{1/2}$
Λ = circumferential lobe number parameter = $4n^2\epsilon^*$
ν = Poisson's ratio
ρ_1 = stress ratio = $\rho_s(\Lambda)^{1/2}$
ρ_s = stress ratio = τ^*/σ_{cl}
σ_{cl} = classical buckling stress for long cylinders subject to the compressive axial stress = $2E\epsilon^*$
τ = shear stress ratio = $\tau^*/n\tau_{cr}$
τ^* = uniformly distributed shear stress
τ_{cr} = critical torsional shear stress for long cylinder = $2E[(\epsilon^*)^3]^{1/2}/(3)^{1/4}$, Ref. 4
ϕ = angle
ϕ^* = angle index = $(\epsilon^*)^{1/2}\phi$
∇^{*2} = Laplacian operator = $\partial^2/\partial_x{}^{*2} + \partial^2/\partial_y{}^{*2}$
∇^2 = Laplacian operator = $\partial^2/\partial_x{}^2 + \partial^2/\partial\phi^2$

Basic Equations

For a long, thin-walled cylindrical shell subjected to torsional shear stress, assumed to be uniformly distributed, the Donnell's equation[1] is

$$\frac{Eh^3}{12(1-\nu^2)}\nabla^{*8}w^* + \frac{Eh}{a}\frac{\partial^4 w^*}{\partial x^{*4}} + 2h\nabla^{*4}\left(\tau^*\frac{\partial^2 w^*}{\partial x^*\partial y^*}\right) = 0 \quad (1)$$

With proper substitution of notations defined in the Nomenclature, Eq. (1) can be expressed as

$$\nabla^8 w + \partial^4 w/\partial x^4 + 4\nabla^4[\rho_s(\partial^2 w/\partial x\partial\phi)] = 0 \quad (2)$$

The solution of Eq. (2) is assumed to be in the form of $w = \bar{W}\exp(p^*x^* + in\phi^*)$. With further substitution of $x^* = ax(\epsilon^*)^{1/2}$ and $\phi^*(\epsilon^*)^{1/2}$, the assumed solution can be rewritten as

$$w = \bar{W}\exp[px + in\phi(\epsilon^*)^{1/2}] \quad (3a)$$

Let

$$\Lambda = 4n^2\epsilon^* \quad (3b)$$

Then with Eq. (3) in Eq. (2), the polynomial of Donnell's equation is obtained

$$[p^2 - (\Lambda/4)]^4 + p^4 + i2\rho_s p\Lambda^{1/2}[p^2 - (\Lambda/4)]^2 = 0 \quad (4)$$

Approximation for Roots

Equation (4) can be replaced by

$$Z^2 + 2\rho_1 Z - p^2 = 0 \quad (5)$$

by making use of the following identities

$$Z = (1/ip)[p^2 - (\Lambda/4)]^2 \qquad \rho_1 = \rho_s\Lambda^{1/2}$$

If $|\rho_1/p| \ll 1$ is assumed, then the binomial expansion of roots of Eq. (5) can be approximated by an exponential series in ρ_1/p. The preceding operation transforms Eq. (4) into

$$\pm ip^2\exp(\pm\rho_1/p) = [p^2 - (\Lambda/4)]^2 \quad (6)$$

For $\rho_1 = 0$, Eq. (6) can be written as

$$p^4 + [p^2 - (\Lambda/4)]^2 = 0 \quad (6a)$$

Two roots of Eq. (6a)[2,3] are

$$\bar{p}_{01} = [1/2(2)^{1/2}]\{-1 - (C+\Lambda)^{1/2} - i[1 + (C-\Lambda)^{1/2}]\} \quad (7a)$$

$$\bar{p}_{02} = [1/2(2)^{1/2}]\{1 - (C+\Lambda)^{1/2} + i[1 - (C-\Lambda)^{1/2}]\} \quad (7b)$$

These two roots have negative real parts and nonvanishing imaginary part for $\Lambda > 0$. Six other distinct roots of Eq. (6a), $\bar{p}_{0j}$ ($j = 3, 4, \ldots, 8$), can thus be obtained for $\Lambda > 0$ by taking complex conjugates and negative of $\bar{p}_{0j}$ and $\bar{p}_{02}$.

For $\rho_1 \neq 0$, the $\bar{p}_{0j}$ are perturbed; a new root p_j is constructed from each $\bar{p}_{0j}$ as

$$p = \bar{p}_0(1 + \alpha\epsilon) \quad (8)$$

where α is a complex number independent of ρ_1. The validity of assuming $|\epsilon|$ small can be shown as follows. By listed definitions

$$\epsilon \equiv \rho_1/\bar{p}_0 \equiv \rho_s\Lambda^{1/2}/\bar{p}_0 \quad (9)$$

For $\Lambda > 0$, Eqs. (7) show that $|\bar{p}_{0j}| > 0$ ($j = 1, 2, \ldots, 8$). Hence $|\epsilon|$ can be made as small as desired by restricting ρ_s to be small. As Λ tends to zero, $|\bar{p}_{01}|$ tends to unity, but $|\bar{p}_{02}|$ behaves as

$$|\bar{p}_{02}| \approx n^2\epsilon^* = \Lambda/4 \quad (10)$$

Hence, for small Λ, in the case of $\bar{p}_{02}$ and similar cases, $|\epsilon|$ behaves as

$$|\epsilon| = 4\rho_s/(\Lambda)^{1/2} = 2\rho_s/\sigma(\epsilon^*)^{1/2} \quad (11)$$

Therefore, the limiting value of $|\epsilon|$ for $\bar{p}_{02}$ as Λ tends to zero, using Eq. (11) and listed definitions for ρ_s, τ_{cr}, and σ_{cl}, is

$$|\epsilon| = [2/n(3)^{1/4}](\tau^*/\tau_{cr}) \quad (12a)$$

It is observed[5] that, for application of Donnell's equation, n must be moderately large. Thus, the coefficient of τ^*/τ_{cr} in Eq. (12a) is less than unity. Besides $(\tau^*/\tau_{cr}) < 1$ is a

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necessary condition for torsional stability. Thus $|\epsilon|$ remains small in the case where Λ tends toward zero. In general ϵ may be written as

$$\epsilon = [(\epsilon^*)^{1/2}/(3)^{1/4}](\Lambda^{1/2}/\bar{p}_0)(\tau^*/\tau_{cr}) \tag{12b}$$

In the case where Λ is of order one, then Eq. (7) shows that $|\Lambda^{1/2}/\bar{p}_0|$ is of order one. Since for thin-walled shells, $(\epsilon^*)^{1/2} \ll 1$, then it follows that, in this case, all $|\epsilon| \ll 1$. Therefore, for thin-walled shells and n moderately large, $|\epsilon| \ll 1$ may reasonably be assumed for all τ^* less than the critical buckling value τ_{cr}. Thus ϵ^2 and higher-order ϵ terms may be neglected for the evaluation of α in Eq. (8).
Let

$$\Lambda/4 \equiv \bar{p}_0{}^2\beta \tag{13}$$

where β is a complex number. Then Eq. (6) can be conveniently rewritten with the help of Eqs. (8, 9, and 13) as

$$\pm i(1 + \alpha\epsilon)^2 \exp(\mp\epsilon/1 + \alpha\epsilon) = \bar{p}_0{}^2[(1 + \alpha\epsilon)^2 - \beta]^2 \tag{14}$$

Rearrangement of Eq. (14) as a linear function of ϵ gives

$$\left(2\alpha \mp 1 - \frac{4\alpha}{1-\beta}\right)\epsilon \pm \left[i\,\frac{\pi}{2} - 2\log\bar{p}_0 - 2\log(1-\beta)\right]\epsilon^{\circ} = 0 \tag{15}$$

Since Eq. (15) is assumed to be true for some range of $0 \leq \epsilon \ll 1$ in which ϵ can take any value, then it follows that Eq. (15) can be established only if the coefficients of ϵ are identically zero. The coefficient of ϵ^0 is automatically satisfied, since $\bar{p}_0$ satisfies Eq. (6) with $\rho_1 = 0$. Both $\bar{p}_{01}$ and $\bar{p}_{02}$ have been defined so as to take the plus (+) sign in the coefficient of ϵ as can be verified by comparing Eq. (6a) with the identity [Eq. (5a) of Ref. 2]

$$[\bar{p}_{0j}{}^2 - (\Lambda/4)]^2 = i\bar{p}_{0j}{}^2 \qquad (j = 1,2) \tag{16}$$

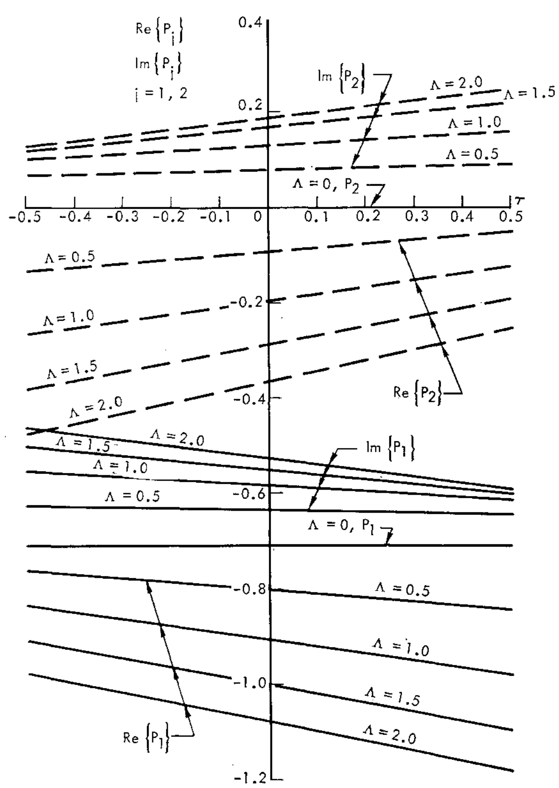

Fig. 1 Approximation to real and imaginary roots of Eq. (4) as function of shear stress ratio τ for various values of circumferential lobe number parameter Λ.

Similarly, when $\rho_1 \neq 0$, α is taken to be

$$\alpha = (\beta - 1)/2(\beta + 1) \tag{17}$$

The combination of Eqs. (8) and (17) yields the required roots, namely,

$$p = \bar{p}_0\{1 + [(\beta - 1)/2(\beta + 1)]\epsilon\} \tag{18}$$

Substitution of the listed definitions of τ and Λ in Eq. (12b) leads to

$$\epsilon = \Lambda\tau/2(3)^{1/4}\bar{p}_0 \tag{19}$$

Therefore, Eq. (18) can be rewritten by making use of Eq. (19) as

$$p = \bar{p}_0 + [(\beta - 1)\Lambda/4(3)^{1/4}(\beta + 1)]\tau \tag{20a}$$

$$= \bar{p}_{0j} + \beta_{0j}\tau \qquad (j = 1, 2, 3, 4) \tag{20b}$$

where

$$\beta_{0j} = (\beta - 1)\Lambda/4(3)^{1/4}(\beta + 1) \tag{21a}$$

or

$$\beta_{0j} = (\Lambda - 4\bar{p}_{0j}{}^2)\Lambda/4(3)^{1/4}(\Lambda + 4\bar{p}_{0j}{}^2) \tag{21b}$$

if Eq. (13) is introduced in Eq. (21a).

The roots of p vs τ are shown in Fig. 1. It is revealed that, for $\Lambda > 0$ and $\tau \geqq 0$, $p_3 = \bar{p}_1$ and $p_4 = \bar{p}_2$. The proof is obvious since τ is real and the conjugate relation holds in Eq. (20b). Finally, the negatives of these four roots complete the eight distinct roots of p_j in Eq. (4).

References

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Review of Recent Developements in Turbulent Supersonic Base Flow

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Introduction

TWO-DIMENSIONAL supersonic base-pressure problems and, in particular, the flow over a backward-facing step have been successfully analyzed, to a certain extent, by various authors.[1–5] The flow model used by most authors has four basic components (the mixing region, the region of confluence, the wake, and the main flow). Of these, the reattachment phenomena in the region of confluence is rather an important one and, in the case of flow over a step, dominates the variation of base pressure. Setting the pressure rise to reattachment equal to the difference between the base pressure and the pressure far downstream, Korst[4] obtained

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